

# How good is $\mu$ - $\tau$ symmetry after results on non-zero $\theta_{13}$ ?

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## Abstract

Viability of the  $\mu$ - $\tau$  interchange symmetry imposed as an approximate symmetry (1) on the neutrino mass matrix  $\mathcal{M}_{\nu f}$  in the flavour basis (2) simultaneously on the charged lepton mass matrix  $M_l$  and the neutrino mass matrix  $M_\nu$  and (3) on the underlying Lagrangian is discussed in the light of recent observation of a non-zero reactor mixing angle  $\theta_{13}$ . In case (1),  $\mu$ - $\tau$  symmetry breaking may be regarded as small (less than 20-30%) only for the inverted or quasidegenerate neutrino mass spectrum and the normal hierarchy would violate it by a large amount. The case (2) is more restrictive and the requirement of relatively small breaking allows only the quasidegenerate spectrum. If neutrinos obtain their masses from the type-I seesaw mechanism then small breaking of the  $\mu$ - $\tau$  symmetry in the underlying Lagrangian may result in a large breaking in  $\mathcal{M}_{\nu f}$  and even the hierarchical neutrino spectrum may also be consistent with mildly broken  $\mu$ - $\tau$  symmetry of the Lagrangian. Neutrinoless double beta decay provides a good means of distinguishing above scenarios. In particular, non-observation of signal in future experiments such as GERDA would rule out scenarios (1) and (2).

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## I. INTRODUCTION

After a conclusive evidence of a non-zero  $\theta_{13}$  by several reactor neutrino experiments [1] disfavoring  $\theta_{13} = 0$  with  $\Delta\chi^2 \approx 100$  in a global analysis [2, 3], it is more meaningful to turn the theoretical search for a symmetry leading to zero  $\theta_{13}$  to a systematic study of effects of perturbations on it or to a search for an alternative symmetry which can predict nonzero  $\theta_{13}$ . Some of the specific symmetries which ensure this are identified in the literature [4]. The effect of perturbations to underlying symmetry giving  $\theta_{13} = 0$  can be studied more generally [5, 6] purely at the phenomenological level. Irrespective of any underlying model, one can define an effective  $Z_2$  symmetry which is both necessary and sufficient for obtaining  $\theta_{13} = 0$  [5]. This is generated by the transformation  $S$  :

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta_{23} & \sin 2\theta_{23} \\ 0 & \sin 2\theta_{23} & -\cos 2\theta_{23} \end{pmatrix}, \quad (1)$$

where  $\theta_{23}$  denotes the atmospheric mixing angle. Invariance of the neutrino mass matrix  $\mathcal{M}_{\nu f}$  in flavour basis under  $S$  leads to vanishing  $\theta_{13}$ . A well-motivated special case of  $S$  is the celebrated  $\mu$ - $\tau$  symmetry [7] which is obtained from Eq. (1) when  $\theta_{23} = \pi/4$  :

$$S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2)$$

We will concentrate here on this specific symmetry and consider two different scenarios. In the first, we assume that  $\mu$ - $\tau$  is an effective symmetry of  $\mathcal{M}_{\nu f}$ . This symmetry may be accidental or a consequence of some other (e.g.  $D_4$  [8]) broken symmetry. In such a situation, the (diagonal) charged lepton mass matrix breaks  $\mu$ - $\tau$  symmetry. In an alternative scenario, we regard  $\mu$ - $\tau$  symmetry as more fundamental and impose it as an approximate symmetry of both the charged lepton and neutrino mass matrix. This can arise from  $\mu$ - $\tau$  symmetry imposed at the Lagrangian level itself. We discuss viability or otherwise of the  $\mu$ - $\tau$  symmetry in both these scenario purely from phenomenological considerations.

## II. APPROXIMATELY $\mu$ - $\tau$ SYMMETRIC $\mathcal{M}_{\nu f}$

To be specific, we define  $\mu$ - $\tau$  symmetry by requiring that the eigenvector of the neutrino mass matrix  $\mathcal{M}_{\nu f}$  in flavour basis corresponding to the heaviest (lightest) mass eigenvalue is given by

$$\begin{pmatrix} 0 \\ \pm \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3)$$

in case of the normal (inverted) hierarchy in the neutrino masses. This requirement leads to the following form for  $\mathcal{M}_{\nu f}$ :

$$\mathcal{M}_{\nu f}^0 = \begin{pmatrix} X & A & \mp A \\ A & B & C \\ \mp A & C & B \end{pmatrix}. \quad (4)$$

These two are special cases of the more general symmetry Eq. (1), obtained when  $\theta_{23} = \pm \frac{\pi}{4}$ . Since sign of  $\theta_{23}$  can be changed by appropriately defining CP violating phases and charged lepton mass eigenstates, it is sufficient to consider only one of the two and we will choose the one corresponding to the negative sign in Eq. (3). All parameters above are complex but two of them say,  $X$  and  $C$  can be made real by redefining the phases of the charged lepton mass eigenstates.  $\mathcal{M}_{\nu f}^0$  is thus characterized by six real parameters and leads to two predictions among eight<sup>1</sup> relevant observables in the neutrino sector. Note that Eq. (3) is an eigenvector of Eq. (4) with the eigenvalue  $B \pm C$ . If this eigenvalue corresponds to the heaviest (lightest) mass eigenstate in case of the normal (inverted) mass hierarchy then the said two predictions are:  $\theta_{13} = 0$  and  $\theta_{23} = \pi/4$ . If this is not the case then one obtains  $\theta_{12} = 0$  or  $\pi/2$  instead of  $\theta_{13} = 0$ . This case is also of interest as a small perturbation to it may result in a large  $\theta_{12}$ , see [9] for a discussion of this case. Here, we only consider the case which predicts  $\theta_{13} = 0$  in the exact  $\mu$ - $\tau$  symmetric limit.

$\mu$ - $\tau$  symmetry is also defined in the literature in a generalized sense which combines ordinary interchange of  $\mu$ - $\tau$  symmetry with the CP [10]. In this case, Eq. (4) gets replaced by

$$\mathcal{M}_{\nu f}^0 = \begin{pmatrix} X & A & A^* \\ A & B & C \\ A^* & C & B^* \end{pmatrix}. \quad (5)$$

Here  $X$  and  $C$  are forced to be real. It is then possible to remove an additional phase from either  $B$  or  $A$  by redefining the charged lepton mass eigenstates without affecting the reality of  $X$  and  $C$ . Above  $\mathcal{M}_{\nu f}^0$  is thus characterized by five real parameters and leads to four predictions among the nine observables. These correspond to two trivial Majorana phases and the relations:

$$\theta_{23} = \frac{\pi}{4} \quad \text{and} \quad \text{Re}(\cos \theta_{12} \sin \theta_{12} \sin \theta_{13} e^{i\delta}) = 0. \quad (6)$$

Unlike in Eq. (4), the  $\mathcal{M}_{\nu f}^0$  as given in Eq. (5) is phenomenologically allowed and the generalized  $\mu$ - $\tau$  symmetry can still remain an exact symmetry. We will concentrate here on the  $\mu$ - $\tau$  symmetry as in Eq. (4) and discuss effect of perturbation on it. The  $\mu$ - $\tau$  symmetry in  $\mathcal{M}_{\nu f}$  implies equalities:  $(\mathcal{M}_{\nu f})_{12} = (\mathcal{M}_{\nu f})_{13}$  and  $(\mathcal{M}_{\nu f})_{22} = (\mathcal{M}_{\nu f})_{33}$ . It is thus natural to characterize its breaking in terms of two complex parameters defined as follows:

$$\epsilon_1 \equiv \frac{(\mathcal{M}_{\nu f})_{12} - (\mathcal{M}_{\nu f})_{13}}{(\mathcal{M}_{\nu f})_{12} + (\mathcal{M}_{\nu f})_{13}}; \quad \epsilon_2 \equiv \frac{(\mathcal{M}_{\nu f})_{22} - (\mathcal{M}_{\nu f})_{33}}{(\mathcal{M}_{\nu f})_{22} + (\mathcal{M}_{\nu f})_{33}}. \quad (7)$$

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<sup>1</sup> Dirac phase  $\delta$  becomes unphysical because of the prediction  $\theta_{13} = 0$ .

We would define approximate  $\mu$ - $\tau$  symmetry as the one in which the absolute values of the above dimensionless parameters  $\ll 1$ . Let us note that

- $\epsilon_{1,2}$  characterize the most general breaking of the  $\mu$ - $\tau$  symmetry and all other elements of an arbitrary perturbation matrix to Eq. (4) can be absorbed in  $\mathcal{M}_{\nu f}^0$  given in Eq. (4).
- One could have normalized the  $\mu$ - $\tau$  breaking denominators in the above equation with a different quantity, *e.g.* the largest neutrino mass. Such a definition would be less conservative and may imply small  $\epsilon_{1,2}$  even when percentage deviation in the differences in the numerator is very large.

One can relate the parameters  $\epsilon_{1,2}$  to observable quantities in a straightforward manner. The most general neutrino mass matrix  $\mathcal{M}_{\nu f}$  can be written after appropriate rephasing of charged lepton mass eigenstates as

$$\mathcal{M}_{\nu f} = U^* \text{Diag.}(m_1, m_2, m_3) U^\dagger, \quad (8)$$

where

$$U = \begin{pmatrix} c_{13}c_{12} & -c_{13}s_{12} & -s_{13}e^{-i\delta} \\ c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} + s_{12}s_{13}s_{23}e^{i\delta} & -s_{23}c_{13} \\ s_{23}s_{12} + c_{12}s_{13}c_{23}e^{i\delta} & c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & & \\ & e^{i\alpha_2/2} & \\ & & e^{i\alpha_3/2} \end{pmatrix}. \quad (9)$$

Here  $c_{ij} = \cos \theta_{ij}$ ,  $s_{ij} = \sin \theta_{ij}$ ;  $\theta_{ij}$  are three mixing angles,  $\delta$  is Dirac CP phase and  $\alpha_{2,3}$  are Majorana CP phases.  $m_{1,2,3}$  are three real and positive neutrino masses. Neutrino (mass)<sup>2</sup> differences  $\Delta_\odot \equiv m_2^2 - m_1^2$  and  $\Delta_A \equiv m_3^2 - m_1^2$  and three mixing angles are now experimentally known. The latest global fit [3] of neutrino oscillation data gives

$$\begin{aligned} \sin^2 \theta_{12} &= 0.30 \pm 0.013 \text{ (0.27 - 0.34)} \Rightarrow \theta_{12} = 33.3^\circ \pm 0.8^\circ \text{ (31}^\circ - 36^\circ) \\ \sin^2 \theta_{23} &= 0.41_{-0.025}^{+0.037} \oplus 0.59_{-0.022}^{+0.021} \text{ (0.34 - 0.67)} \Rightarrow \theta_{23} = 40_{-1.5^\circ}^{+2.1^\circ} \oplus 50.4_{-1.3^\circ}^{+1.2^\circ} \text{ (36}^\circ - 55^\circ) \\ \sin^2 \theta_{13} &= 0.023 \pm 0.0023 \text{ (0.016 - 0.030)} \Rightarrow \theta_{13} = 8.6_{-0.46^\circ}^{+0.44^\circ} \text{ (7.2}^\circ - 9.5^\circ) \\ \frac{\Delta_\odot}{10^{-5}\text{eV}^2} &= 7.5 \pm 0.185 \text{ (7.0 - 8.09)} \quad \text{and} \quad \frac{\Delta_A}{10^{-3}\text{eV}^2} = 2.47_{-0.067}^{+0.069} \text{ (2.27 - 2.69)}, \end{aligned} \quad (10)$$

where (...) denote the  $3\sigma$  ranges of respective observables. The fit obtains two minima for  $\theta_{23}$  and we choose the one corresponding to  $\theta_{23} = 40^\circ$  for the discussions presented in this paper.

It follows from Eqs. (7, 8, 9) that

$$\begin{aligned} \epsilon_1 &= \frac{y + s_{13}f}{1 - s_{13}yf}, \\ \epsilon_2 &= \frac{1}{g_+} ((c_{23}^2 - s_{23}^2)g_- + 4c_{12}s_{12}c_{23}s_{23}s_{13}e^{-i\delta}(-m_1 + m_2e^{-i\alpha_2})) , \end{aligned} \quad (11)$$

with

$$f \equiv \frac{m_3 e^{-i(\alpha_3 - \delta)} - m_1 c_{12}^2 e^{-i\delta} - m_2 s_{12}^2 e^{-i(\alpha_2 + \delta)}}{s_{12} c_{12} (m_1 - m_2 e^{-i\alpha_2})} ,$$

$$g_{\pm} \equiv \pm m_3 e^{-i\alpha_3} c_{13}^2 + m_1 (s_{12}^2 \pm c_{12}^2 s_{13}^2 e^{-2i\delta}) + m_2 e^{-i\alpha_2} (c_{12}^2 \pm s_{12}^2 s_{13}^2 e^{-2i\delta}) . \quad (12)$$

In the above equations,  $s_{13}$  and  $y \equiv (c_{23} - s_{23})/(c_{23} + s_{23})$  or equivalently  $\frac{1}{2}(c_{23}^2 - s_{23}^2) \approx y$  are the  $\mu$ - $\tau$  breaking observables which are small and similar in magnitude, see Eq. (10):  $-0.18 \leq y \leq 0.16$  and  $0.12 \leq s_{13} \leq 0.17$ . Their smallness cannot however be taken as evidence of an underlying approximate  $\mu$ - $\tau$  symmetry due to the presence of functions  $f$  and  $g_{\pm}$  which strongly depend on neutrino mass hierarchy and CP violating phases. They can make  $\epsilon_1$  and/or  $\epsilon_2$  large leading to relatively large breaking of the  $\mu$ - $\tau$  symmetry. It turns out that  $\epsilon_1$  plays a major role in allowing or disallowing  $\mu$ - $\tau$  symmetry and we shall concentrate on it. One could neglect second term in the denominator of  $\epsilon_1$  in Eq. (11) for  $|f| \ll 75$ . In this case,

$$\epsilon_1 \approx y + s_{13} f .$$

Thus one can have small  $\epsilon_1$  for  $|f| \sim \mathcal{O}(1)$ . Let us estimate  $f$  for different neutrino mass hierarchies:

**(A) Normal hierarchy:**  $m_1 \ll m_2 \approx \sqrt{\Delta_{\odot}} \ll m_3 \approx \sqrt{\Delta_A}$

In this case,

$$f \approx -\frac{\sqrt{\Delta_A/\Delta_{\odot}}}{s_{12} c_{12}} e^{i(\alpha_2 - \alpha_3 + \delta)} \left( 1 + \mathcal{O}\left(\frac{\Delta_{\odot}}{\Delta_A}\right) \right) \Rightarrow |f| \approx 12.5 (1 + \mathcal{O}(0.2)) , \quad (13)$$

Such an  $f$  leads to a large  $|\epsilon_1| \approx |y + s_{13} f| \geq 1.5$ . Thus,  $\mathcal{M}_{\nu f}$  cannot be considered to posses an effective  $\mu$ - $\tau$  symmetry if neutrino mass hierarchy is normal.

**(B) Inverted hierarchy:**  $m_1 \approx \sqrt{\Delta_A}$ ,  $m_2 \approx \sqrt{\Delta_{\odot} + \Delta_A} \gg m_3$

The values of  $\epsilon_1$  depend on the CP phases in this case.  $f$  can be approximated as

$$f \approx -e^{-i\delta} \frac{c_{12}^2 + s_{12}^2 e^{-i\alpha_2} + \mathcal{O}(\Delta_{\odot}/\Delta_A)}{s_{12} c_{12} (1 - e^{-i\alpha_2} + \mathcal{O}(\Delta_{\odot}/\Delta_A))} . \quad (14)$$

$|f|$  gets enhanced for  $\alpha_2 \sim 0$  which results in large  $\epsilon_1$  while it is  $\mathcal{O}(\cot 2\theta_{12})$  for  $\alpha_2 \sim \pi$ . Allowed range of  $\alpha_2$  is close to  $\pi/2 < \alpha_2 < \pi$  for which  $|\epsilon_1| \leq 0.2$ .

**(C) Quasi degeneracy:**  $m_1 = m_0 \gg \sqrt{\Delta_{\odot}}$ ,  $m_2 = \sqrt{m_0^2 + \Delta_{\odot}}$ ,  $m_3 = \sqrt{m_0^2 + \Delta_A}$

An idea of allowed values of  $|f|$  can be obtained in this case by considering limiting cases of the Majorana phases corresponding to CP conserving situations. There are four independent possibilities with initial signs of the three masses: (i)  $+++$ , (ii)  $++-$ , (iii)  $-+-$  and (iv)

+ - -. The function  $f$  in these cases is given by

$$f \approx \frac{\pm(1 + \frac{\Delta_A}{2m_0^2})e^{i\delta} - e^{-i\delta}}{-\frac{\Delta_\odot}{2m_0^2}c_{12}s_{12}} \quad \text{for (i) and (ii) ,}$$

$$f \approx \frac{1}{\sin 2\theta_{12}} \left( \pm \left( 1 + \frac{\Delta_A}{2m_0^2} \right) e^{i\delta} - \cos 2\theta_{12} e^{-i\delta} \right) \quad \text{for (iii) and (iv) ,} \quad (15)$$

where positive sign refers to cases (i) and (iii) while negative sign refers to cases (ii) and (iv). It is clear that  $|f|$  is very large  $\geq \Delta_A/\Delta_\odot$  in cases (i) and (ii) while for (iii) and (iv),  $|f| < \cot \theta_{12}$  and maximum value is attained for  $\delta = \pi/2$  (0) in case iii (iv). Both these cases thus allow small  $\epsilon_1$ . In order to find the range of viability of the  $\mu$ - $\tau$  symmetry, one also needs to consider  $\epsilon_2$  and allow non-trivial phases. We show the numerical results of doing this in Fig. 1 which displays the values of  $|\epsilon_1|$  and  $|\epsilon_2|$  as a function of the lightest neutrino mass in case of the normal and inverted hierarchy. We also plot in Fig. 1 the largest contribution to  $\mu$ - $\tau$  breaking, namely  $\text{Max.}\{|\epsilon_1|, |\epsilon_2|\}$ , for a given mass of the lightest neutrino.

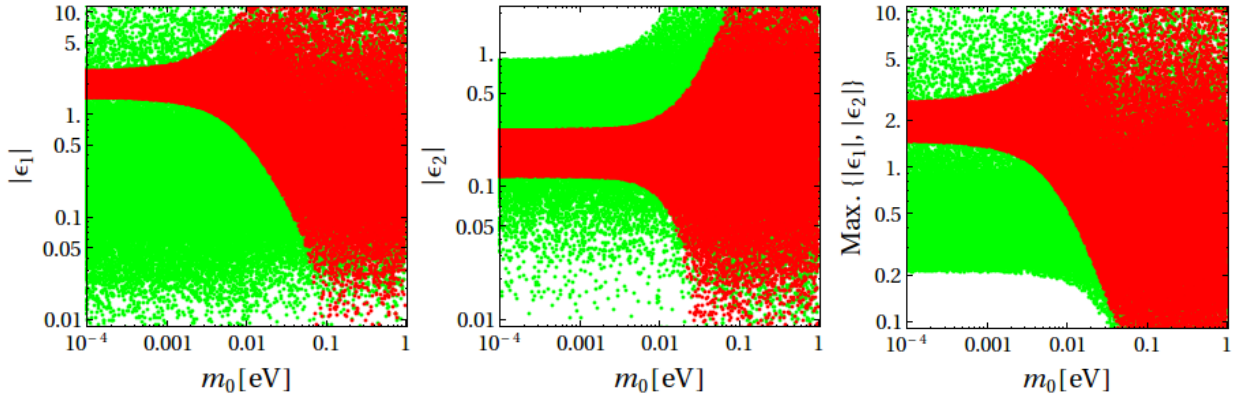


FIG. 1: Allowed values of  $|\epsilon_1|$  (left),  $|\epsilon_2|$  (center) and maximum of  $\{|\epsilon_1|, |\epsilon_2|\}$  (right) as a function of the lightest neutrino mass  $m_0$  in case of the normal (red/dark grey points) and inverted (green/light grey points) hierarchy in neutrino masses. The scattered points are obtained by varying  $\delta$ ,  $\alpha_{2,3} \in [0, 2\pi]$  and for the central values of the other observables as given in Eq. (10).

It is seen from Fig. 1 that the largest contribution to  $\mu$ - $\tau$  breaking comes from  $|\epsilon_1|$  in case of the normal hierarchy. A small violation of  $\mu$ - $\tau$  symmetry, less than 20%, disfavors hierarchical neutrino spectrum ( $m_0 < 0.025$  eV) in this case. The inverted hierarchy allows small values of  $|\epsilon_1|$  or  $|\epsilon_2|$  but both of them are not simultaneously small. In this case, one is able to have small  $\mu$ - $\tau$  breaking, *i.e.*  $\text{Max.}(|\epsilon_1|, |\epsilon_2|) \leq \mathcal{O}(0.2)$  even for  $m_3$  close to zero. Thus, only quasidegenerate or inverted neutrino spectrum provides a viable alternative for the  $\mu$ - $\tau$  symmetry to remain an approximate symmetry of  $\mathcal{M}_{\nu f}$ .<sup>2</sup> This has direct implications in terms of observables namely, the effective mass  $m_{ee}$ , the electron neutrino mass  $m_e$

<sup>2</sup> Similar conclusion has been reached earlier [11] in a specific context of mass matrices with texture zeros.

and sum of the neutrino masses as would be inferred from direct mass determination and cosmology. A small violation of  $\mu$ - $\tau$  symmetry corresponding to  $|\epsilon_{1,2}| \leq 0.3$  leads to the following predictions for these observables:

$$\begin{aligned} |m_{ee}| &\equiv \left| \sum U_{ei}^2 m_i \right| \geq 0.01 \text{ eV}, \\ m_e &\equiv \sqrt{\sum |U_{ei}|^2 m_i^2} \geq 0.02 \text{ eV}, \\ m_{\text{cosmo.}} &\equiv \sum m_i \geq 0.1 \text{ eV}. \end{aligned} \tag{16}$$

Of these, we show the allowed region of  $|m_{ee}|$  as a function of the lightest neutrino mass in Fig. 2. As it can be seen, the region in  $|m_{ee}|$  corresponding to the normal hierarchy

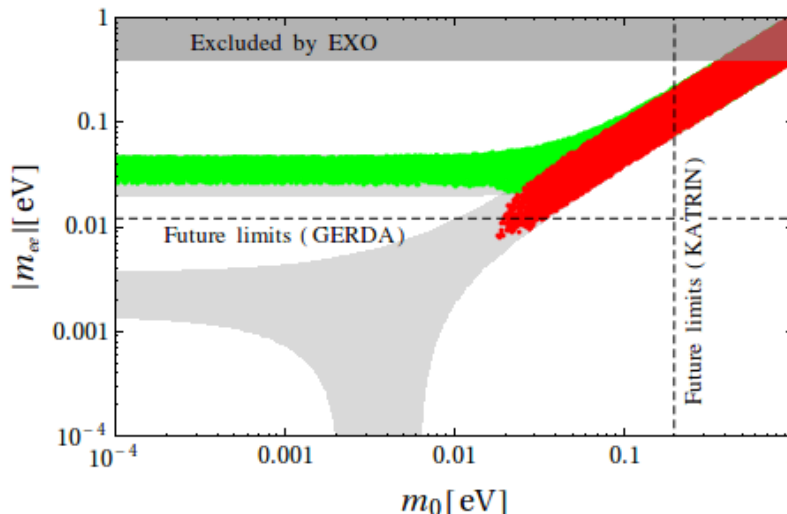


FIG. 2: Allowed ranges of the effective mass  $|m_{ee}|$  as a function of the lightest neutrino mass  $m_0$  in case of the normal (red/dark grey) and inverted (green/light grey) neutrino mass spectrum. The region covered by the scattered points corresponds to  $|\epsilon_1|, |\epsilon_2| \leq 0.3$  while the shaded region corresponds to the most general case without any restriction on  $|\epsilon_{1,2}|$ . The upper grey band shows the region excluded by EXO experiment [12] and the horizontal and vertical dashed lines correspond to the future sensitivity of relevant experiments.

is strongly disfavored if  $\mu$ - $\tau$  symmetry is to remain viable in a way discussed here. In particular, non-observation of signal in experiment like GERDA [13] would practically rule out approximate  $\mu$ - $\tau$  symmetry only of  $\mathcal{M}_{\nu f}$  as a possible explanation behind the small value of  $\theta_{13}$ .

### III. APPROXIMATELY $\mu$ - $\tau$ SYMMETRIC $M_\nu$ AND $M_l$

$\mu$ - $\tau$  symmetry of  $\mathcal{M}_{\nu f}$  need not imply it's presence at the fundamental level. A well-known example is  $A_4$  group imposed as a symmetry of the Lagrangian. This does not even

contain  $\mu$ - $\tau$  symmetry as a subgroup but its spontaneous breaking in a specific manner leads to an  $\mathcal{M}_{\nu f}$  displaying  $\mu$ - $\tau$  symmetry [14]. One could take an alternative point of view and regard  $\mu$ - $\tau$  symmetry itself as more fundamental. We shall now explore phenomenological viability of this scenario. To this end we start by assuming that both the charged lepton mass matrix  $M_l$  and  $M_\nu$  are simultaneously  $\mu$ - $\tau$  symmetric in a suitable basis. More specifically, we assume,

$$\begin{aligned} S_2^T M_\nu S_2 &= M_\nu , \\ S_2^\dagger M_l M_l^\dagger S_2 &= M_l M_l^\dagger , \end{aligned} \quad (17)$$

with  $S_2$  defined in Eq. (2). The  $M_l M_l^\dagger$  is diagonalized by

$$U_l^\dagger M_l M_l^\dagger U_l = D_l \quad (18)$$

$D_l$  is a diagonal matrix and

$$U_l = R_{23}(\pi/4) U_{12} . \quad (19)$$

where  $R_{23}(\pi/4)$  denotes ordinary rotation in the 23 plane by an angle  $\pi/4$  while  $U_{12}$  is a general unitary rotation in the 12 plane. It follows from Eqs. (17, 18) that  $\mathcal{M}_{\nu f} \equiv U_l^T M_\nu U_l$  satisfies

$$\tilde{S}_2^T \mathcal{M}_{\nu f} \tilde{S}_2 = \mathcal{M}_{\nu f} , \quad (20)$$

where

$$\tilde{S}_2 \equiv U_l^\dagger S_2 U_l = \text{Diag.}(1, 1, -1).$$

showing that imposition of the  $\mu$ - $\tau$  symmetry on  $M_l$  and  $M_\nu$  is equivalent to imposing  $\tilde{S}_2$  on  $\mathcal{M}_{\nu f}$ . Thus one should demand  $\mathcal{M}_{\nu f}$  to be invariant under  $\tilde{S}_2$  *and not*  $\mu$ - $\tau$  symmetry if the latter is to arise at the fundamental level. The most general  $\mathcal{M}_{\nu f}$  invariant under  $\tilde{S}_2$  can be written as

$$\mathcal{M}_{\nu f}^0 = \begin{pmatrix} x & a & 0 \\ a & b & 0 \\ 0 & 0 & c \end{pmatrix} . \quad (21)$$

This form and hence the exact  $\tilde{S}_2$  invariance is clearly not a viable proposition since it allows only the solar mixing angle to be non-zero. One must therefore break it. Admitting symmetry breaking,  $\mathcal{M}_{\nu f}$  can be written as

$$\mathcal{M}_{\nu f} = \begin{pmatrix} x & a & \tilde{\epsilon}_1 c \\ a & b & \tilde{\epsilon}_2 c \\ \tilde{\epsilon}_1 c & \tilde{\epsilon}_2 c & c \end{pmatrix} , \quad (22)$$

where  $\tilde{\epsilon}_{1,2}$  parameterize the symmetry breaking. We have normalized them with respect to the (3,3) element of  $\mathcal{M}_{\nu f}$ . We now try to find out under what circumstances  $\tilde{\epsilon}_{1,2}$  can be



small. As before, we express these parameters in terms of observables by comparing Eq. (8) with the form of  $\mathcal{M}_{\nu f}$  given in Eq. (22). This leads to

$$\begin{aligned}\tilde{\epsilon}_1 &= \frac{c_{13}c_{12}s_{12}(m_1 - m_2e^{-i\alpha_2})(-s_{13}c_{23}f + s_{23})}{c_{23}^2g_+ - \cos 2\theta_{23}(m_2e^{-i\alpha_2}c_{12}^2 + m_1s_{12}^2) + \sin 2\theta_{23}c_{12}s_{12}s_{13}e^{-i\delta}(m_1 - m_2e^{-i\alpha_2})}, \\ \tilde{\epsilon}_2 &= \frac{c_{23}s_{23}g_- + \cos 2\theta_{23}c_{12}s_{12}s_{13}e^{-i\delta}(m_1 - m_2e^{-i\alpha_2})}{c_{23}^2g_+ - \cos 2\theta_{23}(m_2e^{-i\alpha_2}c_{12}^2 + m_1s_{12}^2) + \sin 2\theta_{23}c_{12}s_{12}s_{13}e^{-i\delta}(m_1 - m_2e^{-i\alpha_2})}.\end{aligned}\quad (23)$$

where  $f$  and  $g_{\pm}$  are defined in Eq. (12). The magnitudes of these parameters are plotted as a function of the lightest neutrino mass in Fig. 3 for normal and inverted hierarchy. As can

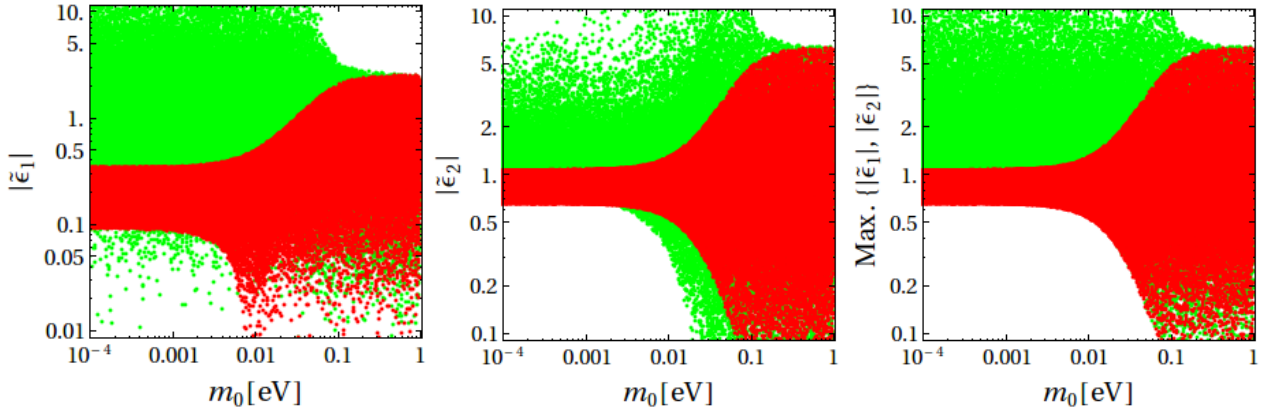


FIG. 3: Allowed values of  $|\tilde{\epsilon}_1|$  (left) and  $|\tilde{\epsilon}_2|$  (center) and maximum of  $\{|\tilde{\epsilon}_1|, |\tilde{\epsilon}_2|\}$  (right) as a function of the lightest neutrino mass  $m_0$  in case of the normal (red/dark grey points) and inverted (green/light grey points) hierarchy in neutrino masses. The scattered points are obtained by varying  $\delta, \alpha_{2,3} \in [0, 2\pi]$  and for the central values of the other observables as given in Eq. (10).

be seen from Fig. 3,

- For both the normal and inverted hierarchies,  $|\tilde{\epsilon}_1|$  and  $|\tilde{\epsilon}_2|$  remain small ( $< 0.2$ ) only if  $m_0 > 0.04$  eV. Thus one cannot regard  $\tilde{S}_2$  as an approximate symmetry of  $\mathcal{M}_{\nu f}$  in these two cases.
- In contrast, for the quasidegenerate spectrum,  $\tilde{S}_2$  and hence  $S_2$  at the fundamental level can be an approximately good symmetry. This was argued earlier in [15] and it can be seen analytically as follows. The diagonalization of Eq. (22) yields in the

approximation of neglecting terms of  $\mathcal{O}(s_{13}^2, as_{13})$  and assuming real parameters,

$$\begin{aligned}
\tan 2\theta_{23} &\approx \frac{2c\tilde{e}_2}{b-c}, \\
\tan 2\theta_{12} &\approx \frac{2(ac_{23} + c\tilde{e}_1s_{23})}{m_2 - x}, \\
\tan 2\theta_{13} &\approx \frac{2(c\tilde{e}_1c_{23} - as_{23})}{m_3 - x}, \\
m_3 &\approx \frac{1}{2} \left( b + c - \frac{b-c}{\cos 2\theta_{23}} \right), \\
m_2 &\approx \frac{1}{2} \left( b + c + \frac{b-c}{\cos 2\theta_{23}} \right), \\
m_1 &\approx \frac{1}{2} \left( x + m_2 + \frac{x - m_2}{\cos 2\theta_{12}} \right).
\end{aligned} \tag{24}$$

As seen from above, a large atmospheric mixing is consistent with a small  $\tilde{e}_2$  for  $b \approx c \gg \tilde{e}_2$  which corresponds to  $m_2 \sim b + c\tilde{e}_2$ ,  $m_3 \sim b - c\tilde{e}_2$ .  $m_1$  is then required to be degenerate if both solar and atmospheric neutrino scales are to be reproduced. In contrast, for  $c \ll b$  or  $b \ll c$ ,  $\tilde{e}_2$  is forced to be  $\mathcal{O}(1)$  and one needs a large  $\mu$ - $\tau$  breaking.

Given these restrictions, it is indeed possible to choose parameters in Eq. (22) which reproduce all the observables correctly. To show this, we try to fit parameters in Eq. (22) in two different ways. In the first, we minimize relevant  $\chi^2$  by restricting  $\tilde{e}_{1,2}$  to be  $\leq 0.1$ . This leads to the following best fit solution

$$\mathcal{M}_{\nu f} = 0.076803 \text{ eV} \begin{pmatrix} 0.8814 & -0.02279 & -0.02223 \\ -0.02279 & 0.9624 & 0.1 \\ -0.02223 & 0.1 & 1 \end{pmatrix}, \tag{25}$$

which reproduce the central values of  $\theta_{23}$ ,  $\theta_{13}$ ,  $\theta_{12}$  and  $\Delta_\odot/\Delta_A$ . The overall mass is normalized to get the correct atmospheric scale. This leads to the following neutrino masses:

$$(m_1, m_2, m_3) = (0.06716, 0.06771, 0.08355) \text{ eV}. \tag{26}$$

Restricting  $\tilde{e}_{1,2}$  to small values automatically leads to quasidegenerate spectrum as would be expected. In contrast, performing the same fit without putting any restrictions on  $\tilde{e}_{1,2}$  led to

$$\mathcal{M}_{\nu f} = 0.03189 \text{ eV} \begin{pmatrix} 0.03992 & 0.2954 & 0.05802 \\ 0.2954 & 0.6988 & 0.6615 \\ 0.05802 & 0.6615 & 1 \end{pmatrix}. \tag{27}$$

This gives correct central values of all observables and neutrino masses

$$(m_1, m_2, m_3) = (0.00388, 0.00948, 0.04985) \text{ eV}. \tag{28}$$

corresponding to a normal spectrum. This however requires large symmetry breaking  $\tilde{e}_2 \sim 0.66$  as would be expected.

#### IV. APPROXIMATE $\mu$ - $\tau$ SYMMETRIC LAGRANGIAN

So far we have assumed mass matrices  $\mathcal{M}_{\nu f}$  alone or  $M_l$  and  $M_\nu$  to be approximately  $\mu$ - $\tau$  symmetric. We now discuss the circumstances under which this symmetry may originate from the symmetry in the underlying Lagrangian. We motivate it through a simple example [15, 16] containing two Higgs doublets  $\phi_{1,2}$ .  $\phi_1$  ( $\phi_2$ ) is assumed even (odd) under the  $\mu$ - $\tau$  symmetry. The Yukawa couplings of the charged leptons then have the following form:

$$-\mathcal{L}_Y = \bar{l}_L(\Gamma_1\phi_1 + \Gamma_2\phi_2)e_R + \text{h.c.} . \quad (29)$$

$l_L, e_R$  respectively denote three generations of the leptonic doublets and singlets. Yukawa couplings  $\Gamma_{1,2}$  satisfy  $S_2^T \Gamma_1 S_2 = \Gamma_1$  and  $S_2^T \Gamma_2 S_2 = -\Gamma_2$ . Approximately  $\mu$ - $\tau$  symmetric  $M_l$  would result from the above if  $|\frac{(\Gamma_2)_{ij}}{(\Gamma_1)_{ij}}| \frac{\langle \phi_1^0 \rangle}{\langle \phi_2^0 \rangle} \ll 1$ . Situations with  $M_\nu$  is however different. If neutrino masses result from the type-II seesaw mechanism with direct coupling of one or more triplet Higgs to neutrinos then just like  $M_l$ ,  $M_\nu$  would also display an approximate  $\mu$ - $\tau$  symmetry. In this case, as shown above  $\mu$ - $\tau$  symmetry can be approximate only for the quasidegenerate spectrum and hierarchical mass spectrum is inconsistent with it. In contrast, if neutrinos obtain their masses from the type-I seesaw mechanism then the Dirac mass matrix  $m_D$  would originate from the Yukawa couplings similar to Eq. (29) and will display an approximate  $\mu$ - $\tau$  symmetry. The explicit Majorana mass matrix  $M_R$  for the right handed neutrinos appears directly in the Lagrangian and would be  $\mu$ - $\tau$  symmetric when this symmetry is imposed on the Lagrangian. The resulting neutrino mass matrix  $M_\nu \approx -m_D M_R^{-1} m_D^T$  may however contain large breaking of the  $\mu$ - $\tau$  symmetry even when  $m_D$  and  $M_R$  are approximately  $\mu$ - $\tau$  symmetric. Thus in type-I seesaw mechanism the normal or inverted hierarchical neutrino spectrum can also be consistent with the approximately  $\mu$ - $\tau$  symmetric Lagrangian. This was realized and discussed in detail in [15]. Here let us illustrate it with a simple but sufficiently realistic example.

Let us assume that  $M_R$  and  $M_l$  are  $\mu$ - $\tau$  symmetric and small breaking of this symmetry occurs only in the Dirac neutrino mass matrix  $m_D$  which is assumed symmetric. The latter is thus parameterized by

$$m_D = \begin{pmatrix} x_D & a_D(1 - \epsilon_{1D}) & a_D(1 + \epsilon_{1D}) \\ a_D(1 - \epsilon_{1D}) & b_D(1 - \epsilon_{2D}) & c_D \\ a_D(1 + \epsilon_{1D}) & c_D & b_D(1 + \epsilon_{2D}) \end{pmatrix} , \quad (30)$$

Here,  $\epsilon_{1D,2D}$  are small  $\mu$ - $\tau$  symmetry breaking parameters. As discussed in [15], if the eigenvalues of  $m_D$  and  $M_R$  are hierarchical and if hierarchy in the right handed neutrino masses are stronger such that the  $M_R$  is nearly singular [17] then the resulting  $M_\nu$  may show large breaking of  $\mu$ - $\tau$  symmetry.  $M_l$  and  $M_R$  are diagonalized by a matrix of the form  $R_{l,R} = R_{23}(\pi/4)R_{12}(\theta_{12l,R})$ . Assuming small  $\theta_{12l,R}$ , neutrino mass matrix in the flavour basis can be written as

$$\mathcal{M}_{\nu f} \approx \tilde{m}_D^T \text{Diag.}(M_1^{-1}, M_2^{-1}, M_3^{-1}) \tilde{m}_D , \quad (31)$$

with  $\tilde{m}_D \equiv R_{23}^T(\pi/4) m_D R_{23}(\pi/4)$ . Comparing this with Eq. (22), one finds to leading order in  $\epsilon_{1D,2D}$ ,

$$\begin{aligned}\tilde{\epsilon}_1 &\approx \frac{\sqrt{2}a_D(\epsilon_{2D}b_DM_1M_3 + \epsilon_{1D}M_2(b_DM_1 - c_DM_1 + M_3x_D))}{(b_D - c_D)^2M_1M_2}, \\ \tilde{\epsilon}_2 &\approx \frac{2\epsilon_{1D}M_2M_3a_D^2 + \epsilon_{2D}b_DM_1(c_D(M_3 - M_2) + b_D(M_2 + M_3))}{(b_D - c_D)^2M_1M_2}.\end{aligned}\quad (32)$$

As shown above, neutrino mass hierarchy  $m_1 \ll m_2 \ll m_3$  requires small  $\tilde{\epsilon}_1$  and relatively large  $\tilde{\epsilon}_2$  (see, Fig. 3). This can be reconciled with a small breaking, *i.e.*  $|\epsilon_{1D,2D}| \ll 1$  at the fundamental level. Let us assume hierarchical eigenvalues  $m_{1D} \ll m_{2D} \ll m_{3D}$  for the Dirac mass matrix  $m_D$ . This can result with  $x_D \ll a_D \sim \sqrt{m_{1D}m_{2D}} \ll b_D, c_D$ . In this case, to the leading order in  $\epsilon_{1D,2D}$  one has  $b_D \approx \frac{1}{2}(m_{2D} + m_{3D})$  and  $c_D \approx \frac{1}{2}(m_{2D} - m_{3D})$ . Inserting these in Eq. (32), one gets

$$\frac{\tilde{\epsilon}_1}{\tilde{\epsilon}_2} \approx \frac{\sqrt{2m_{1D}/m_{2D}}}{1 + \frac{M_2}{M_3} \frac{m_{2D}}{m_{3D}}} \quad \text{and} \quad \tilde{\epsilon}_2 \approx \frac{\epsilon_{2D}}{2} \left(1 + \frac{m_{2D}}{m_{3D}}\right) \left(1 + \frac{m_{2D}M_3}{m_{3D}M_2}\right). \quad (33)$$

We have assumed  $\epsilon_{1D} \ll \epsilon_{2D}$  and neglected contribution of  $\epsilon_{1D}$  in writing the above equation. Strong RH mass hierarchy  $\frac{M_2}{M_3} \ll \frac{m_{2D}}{m_{3D}}$  and hierarchical  $m_{iD}$  automatically lead to enhancement in  $\tilde{\epsilon}_2$  compared to the basic parameter  $\epsilon_{2D}$  and the ratio  $\tilde{\epsilon}_1/\tilde{\epsilon}_2$  remains small as required. As an example,  $\frac{M_2}{M_3} \approx \epsilon_{2D} \frac{m_{2D}}{m_{3D}}$ ,  $m_{1D} \ll m_{2D}$  lead to small  $\tilde{\epsilon}_1$  and  $\tilde{\epsilon}_2 \sim \frac{1}{2}$  as desired.

## V. SUMMARY

We have systematically investigated impact of the measurement of the reactor mixing angle  $\theta_{13}$  on the viability of the  $\mu$ - $\tau$  symmetry. The first investigated scenario is the standard one [7] in which  $\mu$ - $\tau$  symmetry is imposed as an effective symmetry of  $\mathcal{M}_{\nu f}$  only and the charged lepton mass matrix does not respect it. Admitting general symmetry breaking, we found that the symmetry breaking parameters can be small only if the neutrino spectrum is inverted or quasidegenerate. This leads to direct prediction that the neutrinoless double beta decay should be in the observable range. In the second scenario, we assumed both  $M_l$  and neutrino mass matrix  $M_\nu$  to be  $\mu$ - $\tau$  symmetric. This is equivalent to imposing the  $\tilde{S}_2$  symmetry Eq. (20), on  $\mathcal{M}_{\nu f}$ . The diagonal charged lepton mass matrix also remains invariant under this. Again, admitting symmetry breaking, one reaches conclusion that the  $\mu$ - $\tau$  symmetry imposed on  $M_l$ ,  $M_\nu$  is viable as an approximate symmetry only for the quasidegenerate spectrum.

In either scenario, the hierarchical neutrino masses imply large breaking of  $\mu$ - $\tau$  symmetry. If neutrinos obtain their masses from the type-II seesaw mechanism then such large breaking would not allow  $\mu$ - $\tau$  symmetry to be interpreted as a symmetry of the underlying Lagrangian. In contrast, the type-I seesaw mechanism allows interesting possibility in which the required large breaking may be understood as a seesaw amplification of small symmetry breaking in

the underlying Lagrangian. This is illustrated in Section IV and is discussed at length in [15].

To sum up,  $\mu$ - $\tau$  symmetry in either of the presented scenarios is viable as an approximate symmetry in type-II seesaw only if neutrino spectrum is inverted or quasidegenerate in nature. Type-I seesaw mechanism allows also the normal hierarchy in neutrino masses and a small breaking at the fundamental level.

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